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Fractals in Mathematics

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Abstract. In this paper we shall survey two topics concerning the Fractals: nowhere differentiable functions and self-similar sets in Euclidean space such as Cantor set, Koch curve, and Peano curves.

Key words: fractals, nowhere differentiable functions, difference equations, self-similarity.

§ 0. What is a "fractal"?

The terminology "fractal" was created by Mandelbrot in the description of Nature. To quote from his book [30]; "A fractal is by definition a set for which the Hausdorff dimension strictly exceeds the topological dimension." For example, Cantor's ternary set X_C and von Koch's curve X_K are typical fractal sets, since it is known that $\dim_H(X_C)=(\log 2)/(\log 3)>0$ and $\dim_H(X_K)=(\log 4)/(\log 3)>1$ where $\dim_H(X)$ denotes the Hausdorff dimension of a set X.

The notion of "fractal" is surely based on the classical mathematical works done by Cantor, Weierstrass, Peano, Lebesgue, Hausdorff and so on. It is quite surprising that such pathological counter-examples have something to do with certain fields of Natural Science. Thus, it is desirable to clarify the structure of such "singularities".

The measure theory is one of the most powerful mathematical tools to handle fractal sets. See e.g. Rogers [37] and Falconer [8]. However in this paper we shall survey some topics concerning nowhere differentiable functions and fractal sets in Euclidean space. We hope that this approach will make a contribution toward shedding light on the structure of "strange attractors" in dynamical systems.

§ 1. Nowhere Differentiable Functions

The question of the existence of a continuous nowhere differentiable function was settled affirmatively by Weierstrass. Namely he showed that

$$W(x)=\sum_{n\geq 1} a^n \cos(b^n \pi x), \tag{1.1}$$

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where b is an odd integer and $0<a<1$, $ab>1+(3/2)\pi$, at no point has a differential coefficient, either finite, or infinite with a fixed sign. After slight generalizations by a number of writers, Hardy [10] proved that $W(x)$ does not possess a finite differential coefficient at any point in any case in which $0<a<1$, $b>1$ and $ab\geq 1$.

It is now common sense that most continuous functions are nowhere differentiable in the sense of the Baire category (see e.g. Jarnik [18]). However, a concrete example of such functions sometimes leads to interesting speculations.

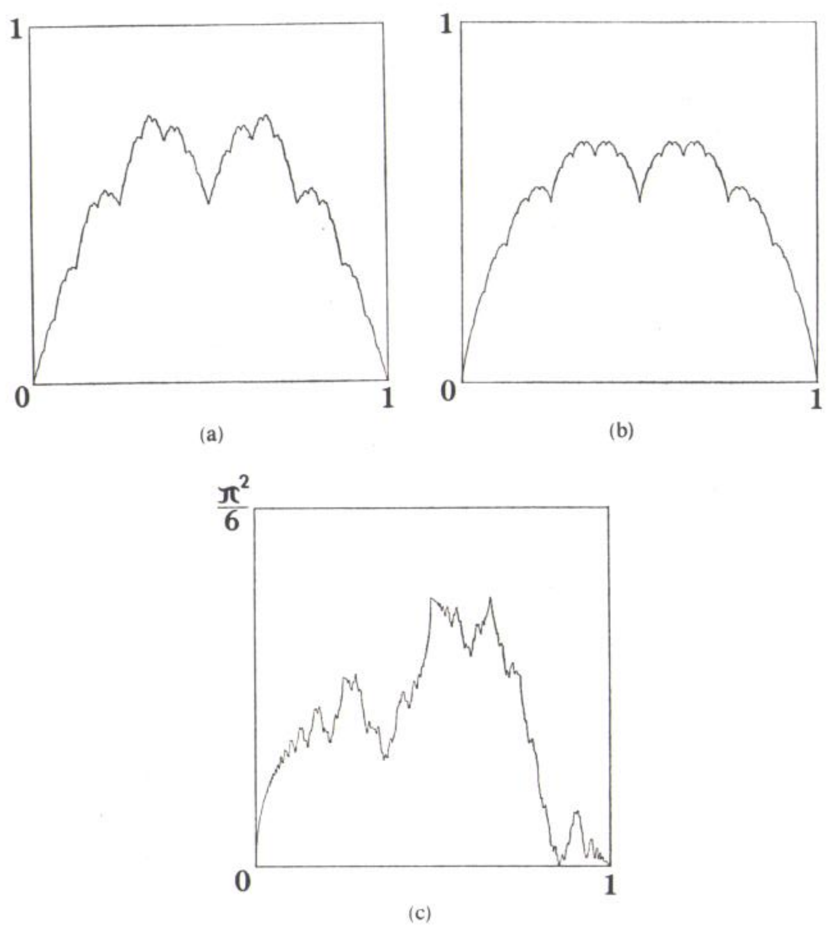


Figure 1. (a) Weierstrass function $(1-W(x))/2$ $(b=a^{-1}=2)$.
(b) Takagi function.
(c) Riemann function.

In 1903, Takagi [40] discovered a quite simple example of a nowhere differentiable function

$$T(x)=\sum_{n\geq 1} 2^{-n}\psi(2^{n-1}x)\,, \tag{1.2}$$

where $\psi(x)=2|x-[x+1/2]|$. This example is highly instructive; that is, $T(x)$ is a typical example of "Condensation of Singularities" (see e.g. Hobson [16, p. 401], since it is a superposition of so-called *saw functions*.

The Weierstrass function (1.1) and the Takagi function (1.2) have cusps at countably many points (see Figure 1(a) and (b)).

Hobson [16, p. 410] also studied the series

$$\sum_{n\geq 1} a^n\psi(b^n x)\,, \qquad 0<a<1\,, \tag{1.3}$$

and showed that the conditions $ab>4$ when b is an even integer or $ab>1$ when b is an odd integer forbid the existence of a differential coefficient finite or infinite, applying a simple method of Knopp. For $b=a^{-1}=10$ this was the example given by van der Waerden [42] in 1930. Also de Rham [35] pointed out that if we take $b=a^{-1}$, b being an even integer, then it has no finite differential coefficient.

Consider now the following functional equation:

$$f(x)-af(bx)=g(x)\,. \tag{1.4}$$

It was de Rham who remarked that the Weierstrass function (1.1) and the series (1.3) satisfy (1.4) for $g(x)=a\cos(b\pi x)$ and $g(x)=a\psi(bx)$ respectively. Kuczma [24, p. 82] noted that if we take $g(x)=a\cos(b\pi x)$, the equation (1.4) has a C^∞ solution in $(-\infty,\infty)$ depending on an arbitrary function, although the *unique* bounded (continuous) solution is the Weierstrass function.

On other non-differentiable functions, Faber [6] considered the function

$$\sum_{n\geq 1} 10^{-n}\psi(2^{n!}x)\,;$$

he showed that this function does not satisfy a Lipschitz condition of any order. Recently Cater [4] studied the function

$$\sum_{n\geq 1} 2^{-n!}\cos(2^{(2n)!}x)\,;$$

he showed that this function has no cusps and satisfies some extreme properties.

It was supposed by Riemann that the function

$$R(x)=\sum_{n\geq 1} n^{-2}\sin(n^2\pi x) \tag{1.5}$$

is nowhere differentiable (see Figure 1(c)). Weierstrass had attempted to prove

Riemann's statement, did not succeed, and was led to the series (1.1). Hardy [10] proved that $R(x)$ has no finite derivative at irrational points nor at rational points of the form $2p/(4q+1)$ or $(2p+1)/(4q+2)$. Gerver [9] proved that $R(x)$ has a derivative $-\pi/2$ at points of the form $(2p+1)/(2q+1)$ and that no finite derivative at points of the form $(2p+1)/2^n$, $n\geq 1$. Finally Smith [39] gave a complete answer to the problem, showing that $R(x)$ has no finite derivative in the remaining cases.

Consider now the following functional equation

$$\frac{1}{p}\left\{f\left(\frac{x}{p}\right)+f\left(\frac{x+1}{p}\right)+\cdots+f\left(\frac{x+p-1}{p}\right)\right\}=\lambda f(\mu x)\ . \tag{1.6}$$

This equation was studied by Artin [1] in characterizing Euler's Gamma function as a unique smooth solution of certain functional equations. The author [11] regarded (1.6) as an eigenvalue problem for some Perron-Frobenius operator and investigated various solutions of (1.6) according to the eigenvalue λ. He also remarked that if $b\geq 2$ is an integer, then the Weierstrass function $W(2x)+\cos(2\pi x)$ satisfies (1.6) for $p=b$, $\mu=1$ and $\lambda=a$; the Takagi function $T(x)-1/2$ also satisfies (1.6) for $p=2$, $\mu=1$ and $\lambda=1/2$; and the Riemann function $R(2x)$ satisfies (1.6) for $p=2$, $\mu=2$ and $\lambda=1/4$.

The Weierstrass function (1.1) is a typical example of a *lacunary series*, that is a series where the terms different from zero are very sparse. More generally, Kaplan, Mallet-Paret and Yorke [20] studied the series

$$f(x)=\sum_{n\geq 1} a^n r(b^n x)\ , \qquad 0<a<1\ , \tag{1.7}$$

where $ab>1$ and $r(x)$ is an almost periodic function. They showed that under certain smoothness conditions on r the series (1.7) is either continuously differentiable or nowhere differentiable; moreover, in the latter case the metric (capacity) dimension of Γ_f is equal to $2+(\log a)/(\log b)$, where Γ_f is the graph of the function (1.7).

It is possible for Γ_g to have the Hausdorff dimension greater than 1 if g is sufficiently singular. Besicovitch and Ursell [2] have shown that if $g(x)$ belongs to the calss Lip(δ), $0<\delta<1$, Γ_g has a finite k-dimensional measure for $k=2-\delta$, and they have constructed g for which the k-dimensional measure is actually positive for $1\leq k\leq 2-\delta$. More generally, Love and Young [29] have shown that if $x(t)$ belongs to the class Lip(δ) and $y(t)$ to the class Lip(δ') where $\delta+\delta'>1$, $0<\delta'\leq\delta\leq 1$, the curve $(x(t), y(t))$ has a finite k-dimensional measure for $k=2-(\delta+\delta'-1)/\delta$. Kline [21] constructed a curve $(x(t), y(t))$ for which the dimension $k=2-(\delta+\delta'-1)/\delta$ is actually attained. Falconer [8] showed that if

$$g(x)=\sum_{n\geq 1}\lambda_n^{-s}\psi(\lambda_n x)\ ,$$

where $0<s<1$ and $\{\lambda_n\}$ is a sequence of positive numbers satisfying

$$\frac{\lambda_{n+1}}{\lambda_n}\to\infty \qquad \text{and} \qquad \frac{\log\lambda_{n+1}}{\log\lambda_n}\to 1 \qquad \text{as} \quad n\to\infty\ ,$$

then $\dim_H(\Gamma_g)=2-s$.

However, it will be difficult to determine the exact value of $\dim_H(\Gamma)$ for the Weierstrass function (1.1) and the series (1.3). We conjecture that in both cases

$$\dim_H(\Gamma)=2+\frac{\log a}{\log b}\ .$$

This value seems to be quite reasonable since Hardy has shown that if $\xi=-(\log a)/(\log b)<1$,

$$W(x+h)-W(x)=O(|h|^\xi) \qquad \text{and} \qquad W(x+h)-W(x)\neq o(|h|^\xi)$$

for any value of x.

§ 2. Chaotic Mappings

Consider a one-dimensional dynamical system $\phi(x)=4x(1-x)$ on the unit interval I. It is well known that the n-fold iteration ϕ^n can be expressed by

$$\phi^n(x)=\sin^2(2^n\arcsin\sqrt{x})\ .$$

It was Prof. Yamaguti who had the inkling to combine ϕ^n with the Weierstrass function (1.1). Indeed, we obtain the fine relation

$$F(a, x)=\sum_{n\geq 0}a^n\phi^n(x)=\frac{1}{2(1-a)}-\frac{1}{2}\sum_{n\geq 0}a^n\cos(2^{n+1}\arcsin\sqrt{x})\ ; \tag{2.1}$$

therefore the generating function $F(a, x)$ is nowhere differentiable for $1/2\leq a<1$. Similarly, as is easily seen, we have

$$F(a, x)=\sum_{n\geq 0}a^n\psi^n(x)=\sum_{n\geq 0}a^n\psi(2^{n-1}x)\ , \qquad \text{for} \quad x\in I\ ; \tag{2.2}$$

therefore $F(1/2, x)$ is nowhere differentiable. See Yamaguti and Hata [44].

Note that $\phi(x)$ is chaotic in the sense of Li-Yorke [27]; moreover there exists a probabilistic invariant measure $dx/(\pi\sqrt{x(1-x)})$, absolutely continuous with respect to the Lebesgue measure. It is also known that $\phi(x)$ is topologically conjugate to the piecewise-linear function $\psi(x)$; that is,

$$H\circ\psi(x)=\phi\circ H(x)\ ,$$

where $H(x)=\sin^2\pi x$ is a homeomorphism of I.

The above examples (2.1) and (2.2) raise the following problem: What kind of function $\omega: I\to I$ causes the non-differentiability of its generating function

$$F(a, x) = \sum_{n\geq 0} a^n \omega^n(x) \tag{2.3}$$

with respect to x? Intuitively, the cause will be the sensitive dependence of initial value for the dynamical system ω. For example, consider a family of quadratic functions $\phi_\lambda(x) = \lambda x(1-x)$ with $0 < \lambda \leq 4$. Then it will be quite interesting to consider the smoothness of its generating function according to the parameter λ. Suppose that there exists a bounded domain D_λ containing an open segment (0, 1) such that $\phi_\lambda(D_\lambda) \subset D_\lambda$. Then it is easily seen that $F(a, z)$ is analytic in D_λ for any $|a| < 1$. For $0 < \lambda < 3$ we may take D_λ to be the interior of the Julia set of the rational function ϕ_λ; in particular, for $\lambda = 2$ we can take D_λ to be an open disk of radius 1/2 centered at $z = 1/2$. On the other hand, for $\lambda = 4$ the Julia set of ϕ_λ is a segment [0, 1] and $F(a, x)$ is nowhere differentiable for $1/2 \leq a < 1$.

Here we should recall the work of Julia [19]. Actually he studied the series

$$J(z) = \sum_{n\geq 0} a^n R^n(z), \qquad |a| < 1,$$

where $R(z)$ is a rational function, and obtained some conditions under which $J(e^{i\theta})$ has no finite derivative with respect to θ; in other words, $J(z)$ has a natural boundary $|z| = 1$. For example, we get the Weierstrass function if we take $R(z) = z^b$.

We now give some criteria for the smoothness of the generating function (2.3).

Theorem 2.1 (Differentiability). *Suppose that ω: $I \to I$ is continuously differentiable and possesses stable periodic points; that is, there exists a point p such that $\omega^q(p) = p$ and $|(\omega^q)'(p)| < 1$. Then the generating function $F(a, x)$ is continuously differentiable in the attractive region W corresponding to $\{p, \omega(p), \cdots, \omega^{q-1}(p)\}$ for any fixed $|a| < 1$.*

Proof. Let K be any compact subset of W and put

$$\mu_n = \sup\{|w'(x)|;\ x \in \omega^{n-1}(K)\} \qquad \text{for} \quad n \geq 1.$$

Since $\omega^n(K)$ converges to the set $\{p, \omega(p), \cdots, \omega^{q-1}(p)\}$ as $n \to \infty$, we have $\limsup_{n\to\infty} \mu_{n+1} \cdots \mu_{n+q} < 1$. This implies the boundedness of $\{\mu_1 \cdots \mu_n\}$ and therefore

$$\sum_{n\geq 0} |a^n(\omega^n)'(x)| \leq \sum_{n\geq 0} |a|^n \mu_1 \cdots \mu_n < \infty \qquad \text{for any} \quad x \in K.$$

Thus the series differentiated term by term converges uniformly and this completes the proof. □

Theorem 2.2 (Non-differentiability). *Suppose that ω: $I \to I$ is continuously*

differentiable and possesses a repulsive fixed point p_0 such that $\lambda \equiv -\omega'(p_0) > 1$. Suppose further that there exists a homoclinic orbit $\{p_{-1} < p_{-3} < \cdots < p_0 < \cdots < p_{-4} < p_{-2}\}$ such that $\omega(p_{-n}) = p_{-n+1}$ for $n \geq 1$ and $p_{-n} \to p_0$ as $n \to \infty$. Then, for any fixed

$$\mu \equiv \sup_{n\geq 1} \frac{|p_{-n-1} - p_0|}{|p_{-n} - p_0|} \leq |a| < 1,$$

the generating function $F(a, x)$ has no finite derivative at any point x for which $\omega^n(x) = p_0$ and $(\omega^n)'(x) \neq 0$ for some $n \geq 0$.

Proof. It suffices to show the non-differentiability at $x = p_0$. We note that $\lambda\mu \geq 1$, since

$$\frac{p_{-n-1} - p_0}{p_{-n} - p_0} = \frac{p_{-n-1} - p_0}{\omega(p_{-n-1}) - p_0} \to \frac{1}{\omega'(p_0)} \qquad \text{as} \quad n \to \infty.$$

Suppose, on the contrary, that there exists a finite derivative $A = (\partial/\partial x)F(a, p_0)$. Then, the equality $F(a, p_{-n-1}) - aF(a, p_{-n}) = P_{-n-1}$ implies $A = (1 + a\lambda)^{-1}$. On the other hand, we have

$$\begin{aligned} \Delta_M &\equiv \frac{F(a, p_{-2M-1}) - F(a, p_0)}{p_{-2M-1} - p_0} \\ &= 1 + a\frac{p_{-2M} - p_0}{p_{-2M-1} - p_0} + \cdots + a^{2M}\frac{p_{-1} - p_0}{p_{-2M-1} - p_0} \\ &= 1 + \sum_{j=1}^{M} \frac{a^{2j-1}}{p_{-2M-1} - p_0}[p_{-2M+2j-2} - p_0 + a(p_{-2M+2j-1} - p_0)]. \end{aligned}$$

First consider the case $a < 0$. Then we have $\Delta_M \geq 1$ since

$$p_{-2M+2j-2} - p_0 + a(p_{-2M+2j-1} - p_0) > 0.$$

Hence $A \geq 1$, contrary to the assumption $A^{-1} = 1 - \lambda|a| \leq 1 - \mu^{-1}|a| \leq 0$. Next, consider the case $a > 0$. Then we also have $\Delta_M \geq 1$ since

$$a \geq \mu \geq \frac{p_{-2M+2j-2} - p_0}{p_0 - p_{-2M+2j-1}}.$$

Hence $A \geq 1$, contrary to $A^{-1} = 1 + a\lambda > 1$. This completes the proof. □

In general, it is a difficult problem to study the differentiability of (2.3) at repulsive periodic points. It is an open problem whether there exists a nowhere differentiable generating function for which the dynamical system ω is not onto.

§ 3. Substitution Operator S_ω

The Weierstrass function (1.1) for $b=2$ can also be represented in the form

$$\sum_{n\geq 0} a^n \cos(2^n \pi x) = \sum_{n\geq 0} a^n \cos(\pi\psi^n(x)) .$$

Thus the Weierstrass function and the series (1.3) corresponding to $b=2$ are particular cases of the following series:

$$F(a, x) = \sum_{n\geq 0} a^n g(\psi^n(x)) , \tag{3.1}$$

where $F(0, x)=g(x)$ is a smooth function on I. It is easily seen that the series (3.1) is a unique continuous solution of the functional equation

$$F(a, x) - aF(a, \psi(x)) = g(x) . \tag{3.2}$$

To deal with the series (3.1), it will be convenient to introduce a substitution operator. Let E be a complex Banach space of all complex-valued continuous functions on I with uniform norm. For a given continuous dynamical system $\omega: I\to I$, we will define the *substitution operator* S_ω by

$$S_\omega(f)(x) = f(\omega(x)) \quad \text{for} \quad x\in I . \tag{3.3}$$

As is easily shown, S_ω is a bounded linear operator of E and its spectrum $\sigma(S_\omega)$ is contained in the unit disk.

It is known that the substitution operator (3.3) is one of the Bourlet operators satisfying a multiplication formula (Targonski [41]). Moreover it is a linear ring endomorphism of our Banach algebra. Note that the eigenvalue problem for the substitution operator leads to the Schröder equation $f(\omega(x))=\lambda f(x)$.

Using the operator S_ψ, the series (3.1) can be written as

$$F(a, x) = \sum_{n\geq 0} a^n S_\psi{}^n(g) = (\mathrm{Id} - aS_\psi)^{-1}(g) ,$$

where the operator $(\mathrm{Id}-aS_\psi)^{-1}$ is known as the resolvent operator of S_ψ. Therefore $(\mathrm{Id}-aS_\psi)^{-1}$ maps $g_0(x)=\cos \pi x$ to the Weierstrass function and $g_1(x)=x$ to the series (1.3) for $b=2$; that is, it maps some smooth functions to nowhere differentiable functions.

If the operator S_ω is completely continuous, then a family on functions $\{\omega, \omega^2, \cdots\}$ must be a compact subset of E. In this respect, we have the following:

Theorem 3.1 ([44]). *Suppose that there exists a sequence* $\{p_{-n}\}_{n\geq 0}$ *such that* $\omega(p_0)=p_0\neq p_{-1}$ *and* $\omega(p_{-n})=p_{-n+1}$ *for* $n\geq 1$. *Then we have* $\sigma(S_\omega)=\{z;\, |z|\leq 1\}$.

By the above theorem, it is possible for a non-chaotic dynamical system ω

to possess the unit disk as a spectrum of S_ω. On the other hand, Bonsall [3] gave an interesting example of S_ω which is completely continuous in a cone C and not in any subspace of E containing C. Let C be a complete positive cone in E consisting of all increasing and convex functions f with $f(0)=0$ and let α be an element of C satisfying $\alpha(1)<1$ and $\alpha'_+(0)>0$. Then the cone map S_α has the desired properties. Note that its partial spectral radius is given by $\alpha'_+(0)$ and there exists an eigenvector $u\in C$ such that $S_\alpha u=\alpha'_+(0)u$.

More generally, we will consider the operator

$$T_\omega(g)(x) = \sum_{n\geq 0} a_n S_\omega{}^n(g) = \sum_{n\geq 0} a_n g(\omega^n(x)) , \tag{3.4}$$

where $\sum a_n$ is an absolutely convergent series. Plainly T_ω is a bounded linear operator with $\|T_\omega\|\leq \sum_{n\geq 0}|a_n|$. By the well known representation theorem, there exists a function $\tau(x, y)$, defined on $I\times I$, satisfying

$$T_\omega(g)(y) = \int_0^1 g(x)d\tau(x, y) ,$$

where $\tau(x, y)$ is of bounded variation with respect to x for each y and is continuous with respect to y as $x=1$. Actually, we can obtain the concrete expression for $\tau(x, y)$ as follows:

$$\tau(x, y) = \sum_{\omega^n(y)\leq x} a_n .$$

On the operator (3.4), we have the following:

Theorem 3.2. *Suppose that a power series* $\sum_{n\geq 0} a_n z^n$ *has a radius of convergence* >1 *and has no roots in the unit disk. Then the operator* T_ω *is a homeomorphism of* E.

Proof. It is clear that the series $\sum_{n\geq 0} b_n z^n = (\sum_{n\geq 0} a_n z^n)^{-1}$ has a radius of convergence >1 and therefore $\sum b_n$ is absolutely convergent. Then, for any $f\in E$, define

$$g(x) = \sum_{n\geq 0} b_n S_\omega{}^n(f) = \sum_{n\geq 0} b_n f(\omega^n(x)) .$$

Hence,

$$\begin{aligned} T_\omega(g) &= \sum_{n\geq 0} a_n S_\omega{}^n(g) = \sum_{n\geq 0}\sum_{m\geq 0} a_n b_m S_\omega{}^{n+m}(f) \\ &= \sum_{l\geq 0}\sum_{n+m=l} a_n b_m S_\omega{}^l(f) = f . \end{aligned}$$

This implies $T_\omega(E)=E$. Next we assume $\sum_{n\geq 0} a_n S_\omega{}^n(g)=0$. Then

$$g = \sum_{l\geq 0}\sum_{n+m=l} a_n b_m S_\omega{}^l(g) = \sum_{m\geq 0} b_m S_\omega{}^m\Big(\sum_{n\geq 0} a_n S_\omega{}^n(g)\Big) \equiv 0 .$$

This implies that T_ω is one to one. Thus, T_ω is a homeomorphism of E. □

Corollary 3.3. *Suppose that a polynomial* $\sum_{n=0}^{N} c_n z^n$ *has no roots in the unit disk. Then the operator* $\sum_{n=0}^{N} c_n S_\omega^n$ *is a homeomorphism of* E.

Note that the conclusion of the above corollary is equivalent to the fact that the linear functional equation

$$c_N g(\omega^N(x)) + \cdots + c_1 g(\omega(x)) + c_0 g(x) = f(x)$$

has a unique solution $g \in E$ for any $f \in E$.

It is also interesting to consider the higher dimensional substitution operator in the form

$$S_{\omega_1,\cdots,\omega_n}(f)(x_1, \cdots, x_n) = f(\omega_1(x_1) + \cdots + \omega_n(x_n)) ,$$

which maps E into the space E_n of all continuous functions defined on the n-dimensional unit cube. In this respect, there is a remarkable result:

Theorem 3.4 (Kolmogorov [22]). *There exists a family of continuous monotone increasing functions* ω_{pq}, *defined on* I, $1 \le p \le n$, $1 \le q \le 2n+1$, *such that the substitution opeartor* S^* *on* E^{2n+1} *defined by*

$$S^*(f_1, \cdots, f_{2n+1}) = \sum_{q=1}^{2n+1} S_{\omega_{1q},\cdots,\omega_{nq}}(f_q)$$

is onto; that is, $S^*(E^{2n+1}) = E_n$.

This is known as the representation theorem of continuous functions of n variables by superposition of continuous functions of one variable and addition.

§ 4. Difference equations

In this section, we will discuss various properties of the function in the form

$$f(x) = \sum_{n \ge 0} c_n \psi^n(x) . \tag{4.1}$$

Obviously the Takagi function (1.2) and the series (2.2) are particular cases of (4.1). First of all, Hata and Yamaguti [14] proved the following theorem using particular orbits of the dynamical system ψ.

Theorem 4.1. *Suppose that the series* (4.1) *converges everywhere. Then the series* $\sum c_n$ *is absolutely convergent.*

Moreover, they showed that the operator L defined by

$$L(\{c_n\}) = \sum_{n \ge 0} c_n \psi^n(x) \in E$$

is a linear homeomorphism from the space of absolutely convergent series onto its image. They also generalized this result to the series (3.4) for $\omega = \psi$.

Faber [7] showed that the series (4.1) has no finite derivative at any point if $\limsup_{n\to\infty} 2^n|c_n| > 0$. This result was accomplished by Kôno as follows:

Theorem 4.2 (Kôno [23]). *The series* (4.1) *has no finite derivative at any point if and only if* $\limsup_{n\to\infty} 2^n|c_n| > 0$. *Moreover, if* $\limsup_{n\to\infty} 2^n c_n = 0$, *it is differentiable on a set of continuum.*

He also studied further properties on the series (4.1). In particular, he showed that the family $\{\psi^n(x) - 1/2\}_{n \ge 0}$ is a concrete example of a multiplicative system but not strongly multiplicative.

In [14], we showed that a continuously twice-differentiable function in the form (4.1) must be a quadratic function. This result was also strengthened by Kôno so that it holds true even in the class of smooth functions in the sense of Zygmund.

Although there are no simple functional equations the series (4.1) must fulfill in general, we can obtain a family of difference equations whose unique continuous solution is the series (4.1). It is convenient to denote the set of lattice points $\{(n, m);\ 0 \le n \le 2^{m-1} - 1,\ m \ge 1\}$ by Ω. Then, the desired equations are

$$f\left(\frac{2n+1}{2^m}\right) - \frac{1}{2}\left\{ f\left(\frac{n}{2^{m-1}}\right) + f\left(\frac{n+1}{2^{m-1}}\right)\right\} = c_m \quad \text{for all} \quad (n, m) \in \Omega \tag{4.2}$$

with boundary conditions $f(0) = 0$ and $f(1) = c_0$. Note that the left hand side of the above equation is essentially the so-called *central difference scheme* for f. Indeed, if we take $c_m = 4^{-m}$, $m \ge 1$, then the equations (4.2) will shift to the differential equation $f'' = -2$, so that $f(x) = c_0 x + x(1-x)$.

Modifying the equations (4.2), consider

$$f\left(\frac{2n+1}{2^m}\right) = (1-\alpha) f\left(\frac{n}{2^{m-1}}\right) + \alpha f\left(\frac{n+1}{2^{m-1}}\right) \quad \text{for all} \quad (n, m) \in \Omega \tag{4.3}$$

with boundary conditions $f(0) = 0$ and $f(1) = 1$ where $0 < \alpha < 1$ is a constant. In [14], we showed that a unique continuous solution of (4.3) satisfies the following functional equation:

$$f(x) = \begin{cases} \alpha f(2x) & \text{for } 0 \le x \le \dfrac{1}{2} , \\ (1-\alpha) f(2x-1) + \alpha & \text{for } \dfrac{1}{2} \le x \le 1 . \end{cases} \tag{4.4}$$

This is a particular case of *de Rham's functional equations*: actually he proved the following

Theorem 4.3 (de Rham [36]). *Suppose that F_0 and F_1 are contractions in* $\mathbf{R}^n$. *Then the functional equation*

$$f(x)=\begin{cases} F_0(f(2x)) & \text{for } 0\le x\le \dfrac{1}{2}, \\ F_1(f(2x-1)) & \text{for } \dfrac{1}{2}\le x\le 1 \end{cases} \tag{4.5}$$

possesses a unique continuous solution if and only if $F_0(p_1)=F_1(p_0)$, where p_0 and p_1 are unique fixed points of F_0 and F_1 respectively.

Moreover, de Rham showed that the solution $L(\alpha, x)$ of (4.4) is strictly monotone increasing and its derivative vanishes almost everywhere if $\alpha\neq 1/2$. Such functions are known as *Lebesgue's singular functions*. The solution $L(\alpha, x)$ was also studied by Lomnicki and Ulam [28] and Salem [38]. It is known that $L(\alpha, x)$ is the distribution function for the Bernoulli trials of unfair coin tossings. In [14], we obtained the following expression

$$L(\alpha, x)=x+\left(\alpha-\frac{1}{2}\right)\sum_{n\ge 0}\sum_{p=0}^{2^n-1}\alpha^{n-m(p)}(1-\alpha)^{m(p)}S_{p,n}(x), \tag{4.6}$$

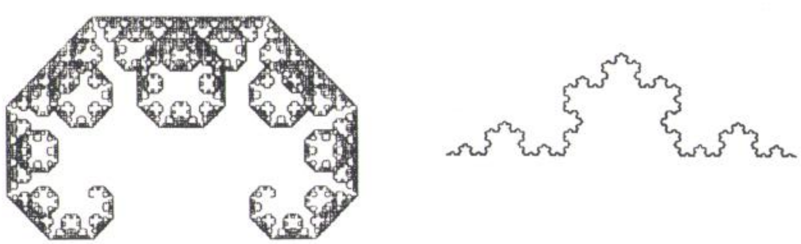
where $m(p)=p-\sum_{n\ge 1}[p/2^n]$ and

$$S_{p,n}(x)=2^n\left\{\left|x-\frac{p}{2^n}\right|+\left|x-\frac{p+1}{2^n}\right|-\left|2x-\frac{2p+1}{2^n}\right|\right\},$$

which is known as the *Schauder base* of E. From this formula, we can obtain a fine relation between the Takagi function (1.2) and the solution of (4.4)

$$\frac{\partial}{\partial\alpha}L\left(\frac{1}{2}, x\right)=2T(x). \tag{4.7}$$

The expression (4.6) is also valid for complex parameter $\alpha\in\{z;\ |z|<1, |1-z|<1\}$ and actually gives a continuous solution of (4.4). In particular,



(a) (b)

Figure 2. (a) Lévy curve.
(b) von Koch curve.

$$L\left(\frac{1}{2}+\frac{i}{2}, x\right)=x+\sum_{n\ge 0}2^{-(n/2)-1}\sum_{p=0}^{2^n-1}S_{p,n}(x)\exp\left[\frac{\pi i}{4}(n+2-2m(p))\right] \tag{4.8}$$

is the curve studied by Lévy [26] (Figure 2(a)). Note that the n-th partial sum of (4.8) gives an approximation broken-line curve.

It is also interesting to consider the following equation instead of (4.4):

$$f(x)=\begin{cases} \alpha\overline{f(2x)} & \text{for } 0\le x\le \dfrac{1}{2}, \\ (1-\alpha)\overline{f(2x-1)}+\alpha & \text{for } \dfrac{1}{2}\le x\le 1. \end{cases} \tag{4.9}$$

De Rham pointed out that the solution of (4.9) becomes the *von Koch curve* for $\alpha=1/2+(\sqrt{3}/6)i$ (Figure 2(b)) and *Pólya's space-filling curve* for $\alpha=1/2+i/2$. The corresponding difference equations to (4.9) are particular cases of the following equations:

$$\begin{cases} R\left(\dfrac{4n+1}{2^{m+1}}\right)=(1-\lambda_m)R\left(\dfrac{n}{2^{m-1}}\right)+\lambda_m R\left(\dfrac{n+1}{2^{m-1}}\right) \\ R\left(\dfrac{4n+3}{2^{m+1}}\right)=\mu_m R\left(\dfrac{n}{2^{m-1}}\right)+(1-\mu_m)R\left(\dfrac{n+1}{2^{m-1}}\right) \end{cases} \tag{4.10}$$

for all $(n, m)\in\Omega$ with conditions $R(0)=0$, $R(1)=1$ and $R(1/2)=\alpha$, where $0<\lambda_m\le\mu_m<1$, $m\ge 1$ are constants. Indeed, if we take $\lambda_m=|\alpha|^2$ and $\mu_m=1-|1-\alpha|^2$, then the continuous solution of (4.10) also satisfies (4.9). It is easily seen that the equations (4.10) possess a unique continuous solution if

$$0<\inf_{n\ge 1}\lambda_n\le\sup_{n\ge 1}\mu_n<1.$$

The curve $R(I)$ is clearly contained in the triangle with vertices 0, 1 and α. $R(I)$ becomes a Jordan curve if $\lambda_n<\mu_n$, $n\ge 1$ and becomes a Peano curve if $\lambda_n=\mu_n$, $n\ge 1$. Also it is easily verified that 2-dimensional Lebesgue measure of the curve $R(I)$ is given by

$$\frac{1}{2}|\mathrm{Im}\,\alpha|\prod_{n\ge 1}(1-\lambda_n-\mu_n).$$

Thus, for a suitable choice of $\{\lambda_n\}$ and $\{\mu_n\}$, we can get a Jordan curve of positive area as a unique continuous solution of (4.10).

§ 5. Self-similar Sets

In this section, we will discuss self-similar sets in Euclidean space $\mathbf{R}^p$. The self-similarity is an important notion in Mandelbrot's book. The Lévy and von Koch curves illustrated in Figure 2 are typical examples of such self-similar sets. It is known that the Lévy curve has a positive 2-dimensional Lebesgue measure and that the Hausdorff dimension of the von Koch curve is given by (log 4)/(log 3); therefore both curves are fractal.

To deal with self-similar fractal sets in $\mathbf{R}^p$, there are at least two methods as far as the author knows. One is accredited to Dekking [5] who used endomorphisms of words in free groups and the other is a method of Hutchinson [17] using a set of contractions; the latter is used in this section.

A mapping F: $\mathbf{R}^p \to \mathbf{R}^p$ is said to be a contraction provided that there exists a constant $\lambda \in (0, 1)$ for which $\|F(x)-F(y)\| \le \lambda \|x-y\|$ for all $x, y \in \mathbf{R}^p$. The least such λ is called the Lipschitz constant of F and denoted by $\mathrm{Lip}(F)$. The unique fixed point of F is denoted by $\mathrm{Fix}(F)$. Then

Definition 5.1 (Hutchinson). A non-void subset X of $\mathbf{R}^p$ is said to be invariant with respect to a set of m contractions $F_1, F_2, \cdots, F_m$ provided that X satisfies the equality

$$X = F_1(X) \cup F_2(X) \cup \cdots \cup F_m(X) . \tag{5.1}$$

This method describing the self-similarity was refound by the author [12] recently. Although Hutchinson's motivation probably has its origin in geometric measure theory, the author studied invariant sets from a general topological point of view. Some results of Hutchinson were strengthened by Mattila [32].

For a set of contractions $F_1, \cdots, F_m$, we can define the mapping

$$\Phi(X) = F_1(X) \cup F_2(X) \cup \cdots \cup F_m(X) \tag{5.2}$$

for an arbitrary subset X of $\mathbf{R}^p$. Obviously the invariant set (5.1) becomes a fixed point of Φ. First of all, we have

Theorem 5.2 (Williams [43], Hutchinson [17]). *For a set of contractions $F_1, \cdots, F_m$, there exists a unique non-void compact invariant set K. Further, for an arbitrary non-void compact subset X of $\mathbf{R}^p$, $\Phi^n(X)$ converges to K in the Hausdorff metric as $n \to \infty$.*

The existence and uniqueness of invariant sets were essentially proven by Williams in 1971 toward a study of generic properties of the action of free (non-abelian) groups on manifolds. The author extended this result for weak contractions.

For example, the Cantor set is a unique compact set invariant under two contractions of $\mathbf{R}$, $F_1(x) = x/3$ and $F_2(x) = (x+2)/3$. The Lévy curve is a unique compact set invariant under two affine contractions of $\mathbf{R}^2$, $F_1(z) = \alpha z$ and $F_2(z) = (1-\alpha)z + \alpha$ for $\alpha = 1/2 + i/2$. Also the von Koch curve is invariant under $F_1(z) = \alpha\bar{z}$ and $F_2(z) = (1-\alpha)\bar{z} + \alpha$ for $\alpha = 1/2 + (\sqrt{3}/6)i$. We will illustrate in Figure 3 some other examples of invariant sets for two affine contractions in $\mathbf{R}^2$. In

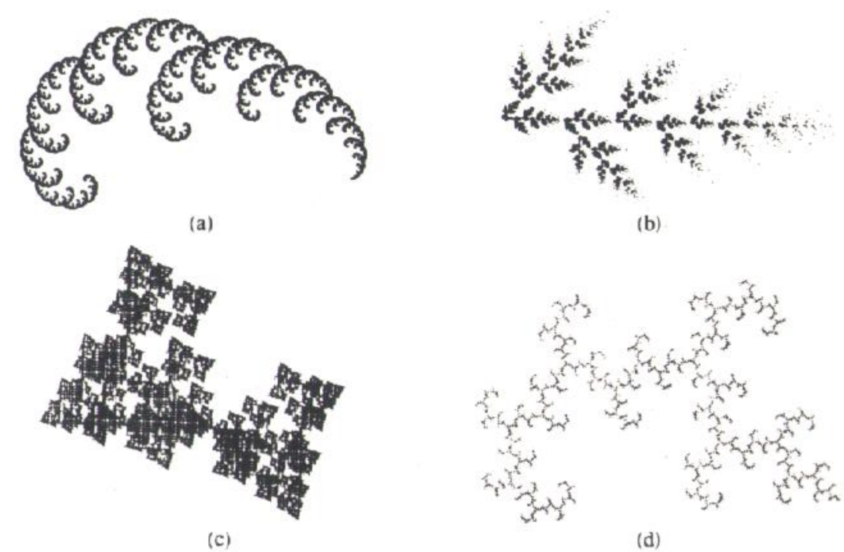


Figure 3. (a) $(0.4614 + 0.4614i,\ 0,\ 0.622 - 0.196i,\ 0)$.
(b) $\left(0,\ \frac{3+3i}{10},\ 0,\ \frac{41}{50}\right)$.
(c) $\left(0,\ \frac{1+i}{2},\ 0,\ \frac{i-1}{2}\right)$.
(d) $(0.4614 + 0.4614i,\ 0,\ 0,\ 0.2896 - 0.585i)$.

all cases, we define

$$F_1(z) = \alpha z + \beta\bar{z} \quad \text{and} \quad F_2(z) = \gamma(z-1) + \delta(\bar{z}-1) + 1 .$$

The corresponding parameters $(\alpha, \beta, \gamma, \delta)$ are given in the captions of Figure 3 respectively.

Modifying the equation (5.1), the author [13] studied the following inhomogeneous equation

$$X = \Phi(X) \cup V = F_1(X) \cup \cdots \cup F_m(X) \cup V , \tag{5.3}$$

where V is a given compact subset of $\mathbf{R}^p$. He proved that there exists a unique non-void compact solution X satisfying (5.3). Moreover, he showed the following analogy to Alternative of Fredholm:

Theorem 5.3. *Suppose that $F_1, \cdots, F_m$ are continuous mappings such that the set $\bigcup_{n \ge 0} \Phi^n(X)$ is pre-compact for any compact X. Then the following statements* (a) *and* (b) *are equivalent;*

(a) *there exists a unique solution of* (5.3) *for every compact V;*
(b) *Φ has a unique fixed point.*

On the Hausdorff dimension of invariant sets we have

Theorem 5.4 (Marion [31], Hutchinson [17]). *Suppose that each contraction F_j, $1 \le j \le m$, is a composition of a dilation, a rotation, a translation and a reflection. Suppose further that there exists a bounded open set U satisfying $\Phi(U) \subset U$ and $F_i(U) \cap F_j(U) = \emptyset$ for $i \ne j$ (The open set condition). Then the s-dimensional Hausdorff measure of the invariant set K is finite and positive; that is,* $\dim_H(K) = s$, *where s is defined by* $\mathrm{Lip}(F_1)^s + \cdots + \mathrm{Lip}(F_m)^s = 1$.

We now turn to the connectedness of invariant sets. First we have

Theorem 5.5 (Williams [43]). *Suppose that* $\mathrm{Lip}(F_1) + \cdots + \mathrm{Lip}(F_m) < 1$ *and that each F_j is injective. Then K is totally disconnected and perfect.*

To study the connectedness of invariant sets, the author [12] introduced the *structure matrix* $M_K = (m_{ij})$ of K as follows:

$$m_{ij} = \begin{cases} 1 & \text{if } F_I(K) \cap F_J(K) = \emptyset \,, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

Theorem 5.6. *The invariant set K is connected if and only if its structure matrix M_K is irreducible. Moreover, if K is connected, it is also a locally connected continuum and arcwise connected.*

If two contractions F_1 and F_2 satisfy $F_1(\mathrm{Fix}(F_2)) = F_2(\mathrm{Fix}(F_1))$, then we can get a parameterization of the invariant set K applying de Rham's Theorem 4.3. In fact, let $f(x)$ be a continuous solution of (4.5). Then,

$$f(I) = f\left(\left[0, \frac{1}{2}\right]\right) \cup f\left(\left[\frac{1}{2}, 1\right]\right) = F_1(f(I)) \cup F_2(f(I)) \,.$$

Therefore $f(I)$ is a compact invariant set under F_1 and F_2, so that $K = f(I)$ as required. In this respect we have

Theorem 5.7. *Let $f(x)$ be a continuous solution of* (4.5). *Then*

(a) *if* $\mathrm{Lip}(F_1) \cdot \mathrm{Lip}(F_2) < 1/4$, *then the Fréchet derivative of f vanishes almost everywhere;*

(b) *if each F_j is a homeomorphism and* $\mathrm{Lip}(F_1^{-1}) \cdot \mathrm{Lip}(F_2^{-1}) < 4$, *then f is not Fréchet differentiable almost everywhere; moreover, if* $\mathrm{Lip}(F_j^{-1}) < 2$ *for* $j = 1, 2$, *then f is nowhere differentiable.*

Note that the above result gives a generalization of Lax's [25] theorem. With these kinds of parameterizations, we can easily get the well known classical Peano curves given by Peano [33], Hilbert [15], and Pólya [34] using certain affine contractions of $\mathbf{R}^2$.

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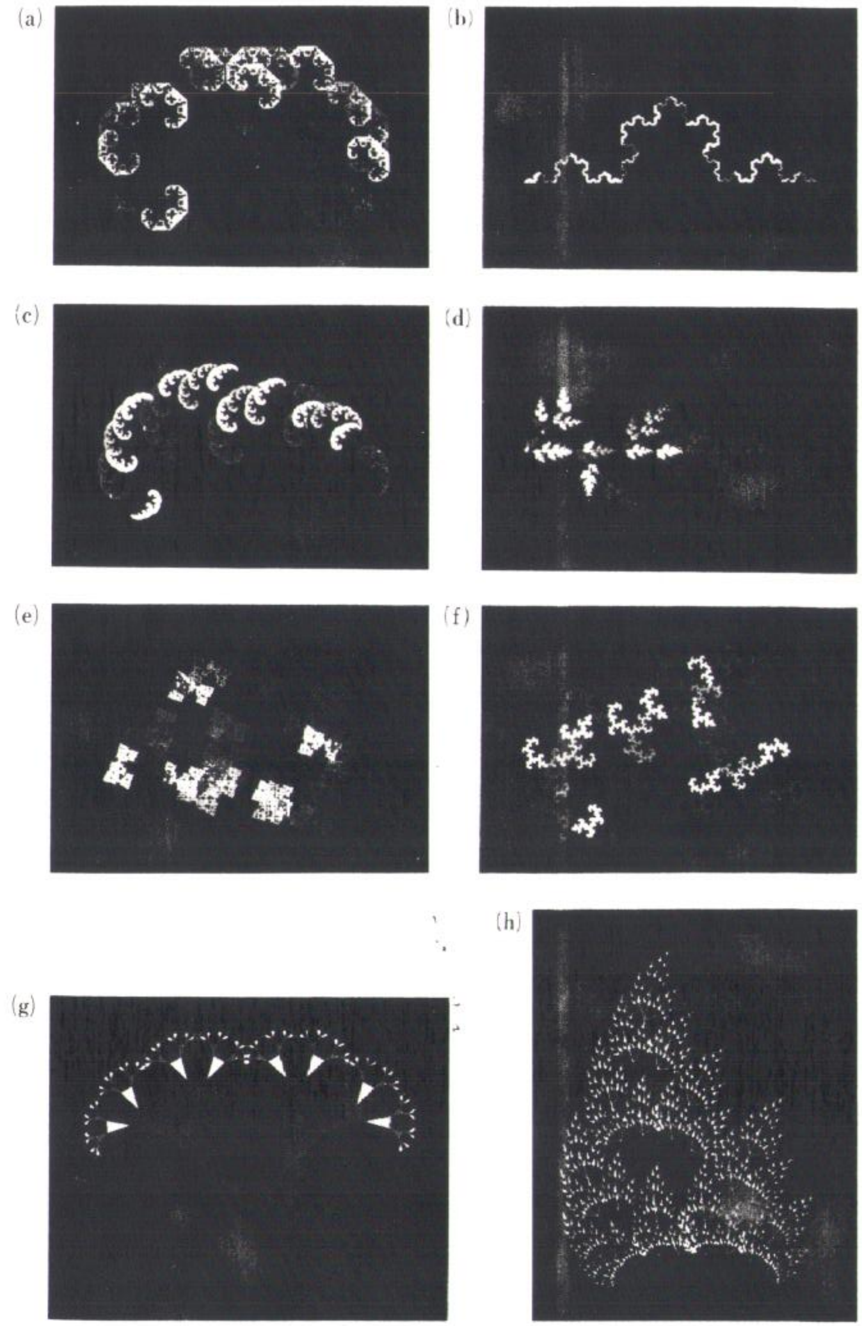

(a) Lévy curve. This is a unique continuous solution of (4.4) for $\alpha=(1+i)/2$ (§ 4).

(b) Von Koch curve. This is a unique continuous solution of (4.9) for $\alpha=1/2+\sqrt{3}i/6$ (§ 4).

(c) Unique invariant set for $F_1(z)=(0.4614+0.4614i)z$ and $F_2(z)=(0.622-0.196i)(z-1)+1$ (§ 5).

(d) Unique invariant set for $F_1(z)=(0.3+0.3i)\bar{z}$ and $F_2(z)=0.82(\bar{z}-1)+1$ (§ 5).

(e) Unique invariant set for $F_1(z)=(0.5+0.5i)\bar{z}$ and $F_2(z)=(-0.5+0.5i)(\bar{z}-1)+1$ (§ 5).

(f) Unique invariant set for $F_1(z)=(0.4614+0.4614i)z$ and $F_2(z)=(0.2896-0.585i)(\bar{z}-1)+1$ (§ 5).

(g) This is a unique solution of (5.3) for $F_1(z)=(0.5+0.6i)z$, $F_2(z)=(0.5-0.6i)(z-1)+1$ and V is the closed triangle with vertices P_0, $F_1(p_0)$ and $F_2(p_0)$ where $p_0=(1-i)/2$.

(h) This is a unique solution of (5.3) for $F_1(z)=(0.5+0.2i)z$, $F_2(z)=(0.5-0.2i)(z-1)+1$, $F_3(z)=(0.6+0.1i)(z-0.5-2i)+0.5+2i$ and V is the union of three segments connecting p_0 with $F_i(p_0)$ for $i=1, 2, 3$, where $p_0=0.5-0.2i$.